Ex: Compute $\iint_R \frac{x-2y}{3x-y}\,dA$ for R the parallelogram given by

$$0 \le x-2y \le 2 \quad \text{and} \quad 1 \le 3x-y \le 8.$$

Sol: We make the substitution $\begin{cases} u = x-2y \\ v = 3x-y \end{cases}$ ← I'm doing this because it makes the region a rectangle!

So $R_{u,v} = \{(u,v) : 0 \le u \le 2,\ 1 \le v \le 8\} = [0,2] \times [1,8]$

under this substitution by the given inequalities for $R_{x,y}$.

We seek Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$, so we need to express $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

$$\begin{cases} u = x-2y \\ v = 3x-y \end{cases} \text{ iff } \begin{cases} x = u+2y \\ y = 3x-v \end{cases} \text{ iff } \begin{cases} x = u+2y \\ y = 3(u+2y)-v \end{cases}$$

$$\text{iff } \begin{cases} x = u+2y \\ -5y = 3u-v \end{cases} \text{ iff } \begin{cases} x = -\frac{1}{5}u + \frac{2}{5}v \\ y = -\frac{3}{5}u + \frac{1}{5}v \end{cases}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{bmatrix} = -\frac{1}{25} - \frac{-6}{25} = \frac{5}{25} = \frac{1}{5}.$$

$$\therefore \iint_{R_{x,y}} \frac{x-2y}{3x-y}\,dA_{x,y} = \iint_{R_{u,v}} \frac{u}{v}\left|\frac{1}{5}\right| dA_{u,v}$$

$$= \frac{1}{5}\int_{v=1}^{8}\int_{u=0}^{2} \frac{u}{v}\,du\,dv$$

$$= \frac{1}{5}\int_{v=1}^{8} \frac{1}{2}\cdot\frac{1}{v}\left[u^2\right]_{u=0}^{2} dv$$

$$= \frac{1}{5}\int_{v=1}^{8} \frac{1}{2}(4-0)\cdot\frac{1}{v}\,dv$$

$$= \frac{2}{5}\left[\ln|v|\right]_{v=1}^{8}$$

$$= \frac{2}{5}\left(\ln(8) - \ln(1)\right) = \frac{2}{5}\left(\ln(2^3) - 0\right)$$

$$= \frac{6}{5}\ln(2)$$

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